Lec 37: 11/23/2009

Translation Invariance (Cont'd):

Recall that for a translationally invariant system we have:

$[H, p] = 0$ (in one dimension)

Therefore H and p have the same eigenstates. Then eigenvalues of p can be used to uniquely label eigenstates of H:

$[H, p] = 0 \Rightarrow V = V_0$

$H|E\rangle = E|E\rangle \quad p|E\rangle = p|E\rangle \quad p = \pm\sqrt{2m(E - V_0)}$

$|E\rangle \propto \exp\left(\frac{ip}{\hbar}x\right)$

Energy eigenstates are doubly degenerate, and p can be used to distinguish positive and negative exponents. However, note that the degenerate eigenstates are not related to each other by a translation. In fact, translating one eigenstates results in:

$x \to x+a \Rightarrow \exp\left(\frac{ipx}{\hbar}\right) \to \exp\left(\frac{ipx}{\hbar}\right)\exp\left(\frac{-ipa}{\hbar}\right)$

The difference is a pure phase, which is irrelevant.

The degenerate states are actually parity transform of one another.

An important point is that for a one particle system translational invariance puts a strong constraint on the potential, i.e. that V is a constant.

For a multi-particle system the situation is different.

Lets consider a two particle system:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(x_1, x_2)$$

Going to the center-of-mass frame we have:

$$X_{CM} = \frac{m_1 X_1 + m_2 X_2}{m_1 + m_2} \qquad X_{rel} = X_1 - X_2$$

$$P_{CM} = (m_1 + m_2) X_{CM} \qquad P_{rel} = P_1 - P_2$$

We then have:

$$H = \frac{p_{CM}^2}{2(m_1+m_2)} + \frac{p_{rel}^2}{2\nu} + V(X_1, X_2) \qquad \nu = \frac{m_1 m_2}{m_1+m_2}$$

Note that $[p_{CM}, X_{rel}] = 0$, hence we have $[p_{CM}, H] = 0$ provided that $V(X_1, X_2) = V(X_1 - X_2)$.

This implies translational invariance for a two particle system if the potential depends only on X_{rel}. This happens, for example, for two charged particle where there is no background electromagnetic field.

Translational invariance therefore puts a much milder constraint on a multi-particle system.

Time Translation Invariance:

Time-translation invariance requires that the Hamiltonian has no explicit time dependence.

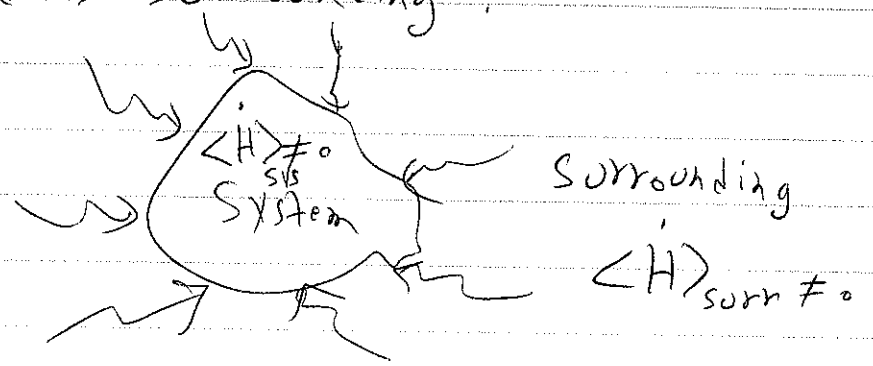
$$\frac{\partial H}{\partial t} = 0$$

This is something we have assumed so far. This

results in;

$\langle \dot{H} \rangle = 0$

Thus the average of energy is a constant. This is always true for an isolated system. $\langle H \rangle$ can change in time if there is interaction between the system and its surrounding;



This, however, happens at the enfense of a change in the average energy of surrounding. The total average energy (system + surrounding) remains constant if the surrounding is sufficiently large enough such that system + surrounding constitutes a larger isolated system.

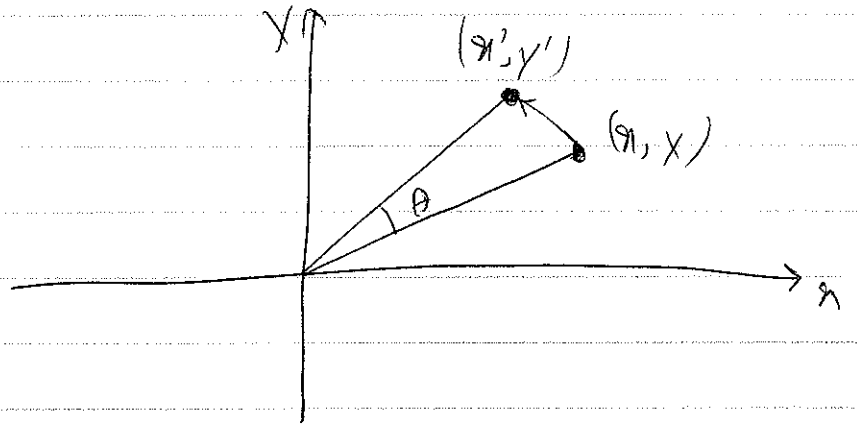
Another important point is that translation invariance and time-translation invariance allow us to describe the same experiment performed at different places and times through the same laws and equations. There would be no law per say if there was no translation invariance or time-translation invariance. In fact, these invariances are not always exact. All that is required is to have translational and/or time-translational invariance in a practical sense. For example, the force between two charged particles depends on their exact location because of various medium effects. However, these additional effects do not matter as long as they are subleading.

Rotation in Two Dimensions:

Rotation by angle θ changes the coordinates of a point (x, y) to (x', y') where;

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$



Rotation can be clockwise or counter-clockwise.

Rotation is related to angular momentum. In two dimensions angular momentum is given by;

$$l_z = x p_y - y p_x$$

We keep the subscript z, even though there is no third dimension, for later generalization to three dimensions. In quantum mechanics angular momentum is an operator;

$$L_z = X P_y - Y P_x$$

It is easy to verify the following commutation relations;

$[X, L_z] = -i\hbar Y \qquad [Y, L_z] = i\hbar X$

$[p_x, L_z] = -i\hbar p_y \qquad [p_y, L_z] = i\hbar p_x$

Consider a function $\psi(x,y)$. Under rotation by angle θ we have:

$$\psi(x,y) \to \psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$$

For a very small angle $\theta \ll 1$ we have $\cos\theta \simeq 1$ and $\sin\theta \simeq \theta$. Thus:

$$\psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) \simeq \psi(x + \theta y, y - \theta x)$$

$$= \psi(x,y) + \frac{\partial\psi}{\partial x}(x,y)\,\Delta x + \frac{\partial\psi}{\partial y}(x,y)\,\Delta y + \text{higher\_order terms}$$

$$\simeq \psi(x,y) + \theta y \frac{\partial\psi}{\partial x} - \theta x \frac{\partial\psi}{\partial y}$$

Note that in the position basis:

$$L_z = -i\hbar x \frac{\partial}{\partial y} + i\hbar y \frac{\partial}{\partial x}$$

Hence:

$$\psi(x,y) \to \psi(x,y) - \frac{i}{\hbar}\theta L_z \psi(x,y) = \left(I - \frac{i}{\hbar}\theta L_z\right)\psi(x,y)$$

This is valid for $\theta \ll 1$. For a finite angle, it can

be shown that;

$\Psi(x,y) \longrightarrow R(\theta)\,\Psi(x,y) \qquad R(\theta) = \exp\left(\frac{-i\theta L_z}{\hbar}\right)$

$R(\theta)$ is the rotation operator, operating on a vector $|\Psi\rangle$ in the Hilbert space. Angular momentum operator L_z is the generator of rotation.

A system is rotationally invariant if $[H, L_z] = 0$.

For a single particle;

$H = \frac{p_x^2 + p_y^2}{2m} + V(x,y)$

Going to polar coordinates (s,ϕ) such that;

$s = \sqrt{x^2+y^2}\,, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$

We have (in the position basis);

$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s^2}\frac{\partial^2}{\partial \phi^2}\right) + V(s,\phi)$

It can be shown that;

$L_z = -i\hbar\frac{\partial}{\partial \phi}$

It is straightforward to see that:

$$[H, L_z] = 0 \iff V(\rho, \phi) = V(\rho)$$

This is a much weaker constraint than what translational invariance puts on the potential, namely constant potential.

If $V = V(\rho)$, it is called a central potential (e.g., Coulomb potential)

In this case the energy eigenstates are separable:

$$-\frac{\hbar^2}{2\nu}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)\psi_E(\rho,\phi) + V(\rho)\psi_E(\rho,\phi) = E\psi_E(\rho,\phi)$$

(ν we use "ν" for mass to avoid confusion with L_z eigenvalues)

$$\psi_E(\rho,\phi) = R_E(\rho)\,\Phi(\phi)$$

The second order partial differential equation is then reduced to two second order ordinary differential equations:

$$\frac{d^2}{d\phi^2}\Phi_m(\phi) = -m^2\Phi_m(\phi)$$

$$-\frac{\hbar^2}{2\nu}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2}\right)R_{Em}(\rho) + V(\rho)R_{Em}(\rho) = ER_{Em}(\rho)$$

Note that Φ and R are labeled by quantum numbers that enter the corresponding differential equation.